NASA CONTRACTOR REPORT

NASA CR-283

APPROXIMATE METHODS FOR THE COMPUTATION OF WAVE PROPAGATION IN NONUNIFORM MEDIA

by H. Y. Yee

Prepared under Grant No. NsG-608 *by*
UNIVERSITY OF ALABAMA
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for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • AUGUST 1965

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Summary

Applications of the collocational method to wave propagation through a nonuniform region with variation in only one spatial coordinate are shown for plane and cylindrical cases. Scattering and radiation, in the absence of and in the presence of a similar shaped conducting object are formulated. A simple example shows the accuracy of this method. In cases where the nonuniform region varies in more than one spatial coordinate, Green's function is applied to formulate an integral equation. Solutions of the integral equation can be obtained by an iterative method for small variations.

I. Introduction

During the re-entry phase of a space vehicle, a region of ionized air, called plasma, around the vehicle is formed between the body and the shock wave. The communication from the vehicle to ground and vice versa is then affected by the interaction of free electrons with electromagnetic waves in this region. If the ionization is sufficiently strong, a "blackout" of radio transmission may occur.

From the macroscopic point of view, the effect of the interactions between free electrons and electromagnetic waves in a region of plasma in the absence of static magnetic fields can be represented as an isotropic lossy dielectric[1] with permeability $\mu = \mu_o$, and permittivity

$$\epsilon = \epsilon_o \left[1 - \omega_p^2/(\nu^2 + \omega^2) + j\omega_p^2 \nu/\omega(\nu^2 + \omega^2) \right], \qquad (1)$$

where ω = operating angular frequency of electromagnetic wave,

ν = collision frequency of particles in plasma,

ω_p = $(e^2 N/m \epsilon_o)^{1/2}$ plasma frequency,

N = electron density,

e, m = charge and mass of an electron respectively,

μ_o, ϵ_o = permeability and permittivity of vacuum.

In considering electromagnetic wave propagation, the plasma can be treated as a lossy dielectric with permittivity given by Eq. (1). In general, the electron density and collision frequency are functions of spatial coordinates, particularly in the direction normal to the body of the vehicle.[2] Electromagnetic wave propagation in a nonuniform lossy dielectric is essentially the problem of re-entry communication. Of course, the geometry is another important parameter.

Wave propagation in nonuniform media can be dealt with by solving the wave equations derived from Maxwell's equations. Approximation techniques, for instance the WKB method and Born approximation have been applied to this problem.[2,3,4,5] However, all these methods are valid only for small variations in permittivity. Literature[6,7] concerning numerical solutions is available, but analytic solutions are always more desirable.

Consider first the case where the permittivity is a real function (for a typical re-entering vehicle, the imaginary part is of the order 10^3 smaller than the real part at 5-10 KMC) of only one spatial coordinate is considered first. Wave propagation in a nonuniform dielectric slab and a nonuniform dielectric cylinder under different boundary conditions are treated separately. The second order differential equation of each case is solved approximately by the method of collocation which converts the differential equation into a system of algebraic equations. As an example to demonstrate the accuracy of this method, the normal scattering of a plane wave by an infinite slab of exponentially varying permittivity is considered. The application of Green's function to wave propagation in nonuniform medium is described also. The integral equation obtained under this consideration can be solved approximately by the method of collocation. It is more convenient to solve the integral equation by Born approximation or iterative method if the variation of the permittivity is small. In this treatment, the variation can be in three spatial coordinates and the permittivity may be complex.

2. Method of Collocation.

If the collision frequency is very small in comparison with the operating frequency, the imaginary term of Eq. (1) can be neglected provided that the real part is never vanishingly small. The problem is confined to simple cases where the permittivity ϵ is a function of only one spatial dependent variable. The method of collocation[8,9] is applicable to achieve approximate solutions for the second order homogeneous differential equations which result from the Maxwell's equations. This method has three advantages:

(1) There is no limitation on the variation of permittivity as long as it is a well behaved function.

(2) Accurate solutions can be achieved even when values of permittivity are known (by experiment) only at a sufficient number of points in space.

(3) A closed form approximate expression for fields within the nonuniform region can be obtained.

Four cases will be considered separately as follows:

a. Propagation of plane wave through an infinite dielectric slab.

Consider the oblique incidence of a plane wave on a plane dielectric slab of thickness a as shown in Figure 1. The perpendicular polarization implies that the electric field vector is perpendicular to the plane of incidence (see Figure 1 (a)), and parallel polarization indicates that the electric field vector is in the plane of incidence as in Figure 1 (b). Suppose that the normalized incident fields for perpendicular polarization and parallel polarization are given by

$$E_{1y}^{i} = \exp.(jk_x x + jk_z z),$$
$$H_{2y}^{i} = Z_0^{-1} \exp.(jk_x x + jk_z z)$$

respectively, where $Z_0^2 = \mu_0 \epsilon_0$. The subscripts 1 and 2 are used to indicate that the quantities are related to the perpendicular and parallel polarization respectively. The propagation constants in the x- and z-directions are given by

$$k_x = k_0 \cos\alpha,$$
$$k_z = k_0 \sin\alpha,$$

where $k_0^2 = \omega^2 \mu_0 \epsilon_0$ and the angle α is the angle of incidence. The total fields in regions I and III (see Figure 1) can be written as

$$E_{1y}^{\mathrm{I}} = \exp.(jk_x x + jk_z z) + R_1 \exp.(-jk_x x + jk_z z),$$
$$E_{1y}^{\mathrm{III}} = T_1 \exp.(jk_x x + jk_z z),$$
$$H_{2y}^{\mathrm{I}} = Z_0^{-1}[\exp.(jk_x x + jk_z z) - R_2 \exp.(-jk_x x + jk_z z)],$$
$$H_{2y}^{\mathrm{III}} = (T_2/Z_0) \exp.(jk_x x + jk_z z),$$

where the superscript denotes the region number, the parameter T and R are the transmission and reflection coefficients respectively. Writing the expressions for the fields inside the dielectric slab, equating the tangential field components at two slab surfaces, and solving for T and R yields

$$T_1 = (2j/k_x D_1) \left[F'_1(o)\, G_1(o) - F_1(o)\, G'_1(o) \right] \exp(jk_x a), \tag{2}$$

$$\begin{aligned} R_1 = D_1^{-1} \{ & F_1(o)\, G_1(a) - F_1(a)\, G_1(o) - k_x^{-2} \left[F'_1(o)\, G'_1(a) - F'_1(a)\, G'_1(o) \right] \\ & + jk_x^{-1} \left[F'_1(o)\, G_1(a) + F_1(o)\, G'_1(a) - F'_1(a)\, G_1(o) - F_1(a)\, G'_1(o) \right] \} \exp.(2jk_x a), \end{aligned} \tag{3}$$

$$T_2 = \left[2j/k_x \epsilon_r(o)\, D_2 \right] \left[F'_2(o)\, G_2(o) - F_2(o)\, G'_2(o) \right] \exp.\,(jk_x a), \tag{4}$$

$$\begin{aligned} R_2 = D_2^{-1} \{ & F_2(a)\, G_2(o) - F_2(o)\, G_2(a) + \left[F'_2(o)\, G'_2(a) - F'_2(a)\, G'_2(o) \right] \left[k_x^2\, \epsilon_r(o)\, \epsilon_r(a) \right]^{-1} \\ & + jk_x^{-1} \left[(F_2(a)\, G'_2(o) - F'_2(o)\, G_2(a)) / \epsilon_r(o) \right. \\ & \left. + (F'_2(a)\, G_2(o) - F_2(o)\, G'_2(a)) / \epsilon_r(a) \right] \} \exp(2jk_x a). \end{aligned} \tag{5}$$

where

$$\begin{aligned} D_1 = {} & F_1(o)\, G_1(a) - F_1(a)\, G_1(o) + k_x^{-2} \left[F'_1(o)\, G'_1(a) - F'_1(a)\, G'_1(o) \right] \\ & + jk_x^{-1} \left[F'_1(o)\, G_1(a) - F_1(o)\, G'_1(a) + F'_1(a)\, G_1(o) - F_1(a)\, G'_1(o) \right], \end{aligned}$$

$$\begin{aligned} D_2 = {} & F_2(o)\, G_2(a) - F_2(a)\, G_2(o) + \left[k_x^2\, \epsilon_r(o)\, \epsilon_r(a) \right]^{-1} \left[F'_2(o)\, G'_2(a) - F'_2(a)\, G'_2(o) \right] \\ & + jk_x^{-1} \{ \left[F'_2(o)\, G_2(a) - F_2(a)\, G'_2(o) \right] / \epsilon_r(o) \\ & + \left[F'_2(a)\, G_2(o) - F_2(o)\, G'_2(a) \right] / \epsilon_r(a) \}. \end{aligned}$$

The derivatives of the function with respect to the argument are denoted by primes. F_1 and G_1 are two linearly independent particular solutions of the following differential equations.[3,8]

$$\psi''(x) + k_o^2 [\epsilon_r(x) - \sin^2\alpha] \psi(x) = 0, \tag{6}$$

where the relative permittivity ϵ_r is a function of x only. The corresponding function F_2 and G_2 are linearly independent and satisfy the differential equation

$$\phi''(x) - [\ln \epsilon_r(x)]' \phi'(x) + k_o^2 [\epsilon_r(x) - \sin^2\alpha] \phi(x) = 0. \tag{7}$$

Obviously Eqs. (6) and (7) are of the same type and can be represented by

$$U''(x) + p(x) U'(x) + q(x) U(x) = 0. \tag{8}$$

Provided that p (x) and q (x) are regular functions within the region under consideration, i.e., p(x) and q(x) have no singularities within the interval $0 \le x \le a$, the two linearly independent particular solutions of Eq. (8) can be approximately expressed by[8]

$$U_e(x) = \sum_{n=0}^{N} A_n \cos L_{en} x, \qquad \text{even function} \tag{9}$$

$$U_o(x) = \sum_{m=1}^{M} B_m \sin L_{om} x. \qquad \text{odd function} \tag{10}$$

where $L_{en} = n\pi / v_e a$, $L_{om} = m\pi / v_o a$. M and N are integers. The two functions U_e and U_o are valid within $0 \le x \le a$. The dimensionless quantities v_e and v_o, to be determined by the differential equation, are two real numbers greater than or equal to unity. They become equal as M and N approach infinity.

The odd solution only is considered here. The even solution can be obtained by the same procedure. Substituting Eq. (10) into Eq. (8) yields

$$\sum_{m=1}^{M} \left\{ \left[q(x) - L_{om}^2 \right] \sin L_{om} x + p(x) L_{om} \cos L_{om} x \right\} B_m = 0. \tag{11}$$

In order to have Eqs. (10) satisfy Eq. (8), Eq. (11) should hold at all points within the interval $0 \leq x \leq a$. But, for the purpose of approximation, the method of collocation[9] requires the equality to be fulfilled only at M points. Let these points be $0 \leq x_1 < x_2 < x_3 < \ldots < x_m = a$. There are many allowable choices of points. Usually, it is convenient to choose equal intervals between points. For each point, Eq. (11) is a linear homogeneous algebraic equation of M unknowns B_m and the parameter ν_o. Hence, a system of M algebraic homogeneous equations with M unknowns and one parameter is then formed, the rest of the work is devoted to solve the eigenvalue and eigenvector problem, that is

$$[D_{im}][B_m] = 0, \tag{12}$$

where $D_{im} = [q(x_i) - L_{om}^2] \sin L_{om} x_i + p(x_i) L_{om} \cos L_{om} x_i.$

The value of ν_o is determined by

$$\det. | D_{im} | = 0. \tag{13}$$

There are many roots of ν_o in Eq. (13). Taking convergence into account, the desirable value is the smallest root which is greater than or equal to unity. With the known value of ν_o the expansion coefficients B_m can be calculated from Eq. (12) in terms of B_r, which is the largest among the B_m's. In fact, the index r indicates r variations of the function $U_o(x)$ within the interval $0 \leq x \leq a$.

Similar procedures lead to solutions of ν_c and the corresponding expansion coefficients A_n of Eq. (9). By this method, the general solutions of Eqs. (6) and (7) are found for a specific frequency within the specified region, and hence the reflection and transmission coefficients can be calculated by Eqs. (3) - (5).

It should be mentioned that the method of least squares[8,9] is also applicable to this case. Multiplying Eq. (11) by $\sin L_{os} x$ and integrating from 0 to a with respect to x yields

$$\sum_{m=1}^{M} B_m \int_0^a \sin L_{os} x \left\{ \left[q(x) - L_{om}^2 \right] \sin L_{om} x + p(x) L_{om} \cos L_{om} x \right\} dx = 0. \tag{14}$$

The integrals in Eq. (14) can be approximated by a weighted sum of the relevant ordinates at k points. That is

$$\sum_{m=1}^{M} B_m \sum_{i=1}^{k} W_s \sin L_{os} x_i \left\{ \left[q(x_i) - L_{om}^2 \right] \sin L_{om} x_i + p(x_i) L_{om} \cos L_{om} x_i \right\} = 0, \tag{15}$$

where s = 1, 2, 3, ... M, and $x_i = ia/k$. The weighting coefficients W_s are conventionally given by either the Trapezoidal rule[10] or Simpson's one-third rule[10]. Of course, other rules can be used as well. Eq. (15) again, is a system of linear, homogeneous algebraic equations which can be explored to find the suitable values for ν_c and B_m's. The solutions obtained by the method of collocation will differ with different choice of the points where Eq. (11) is satisfied. This phenomenon does not exist in the method of least squares which is considerably more accurate, but also more complicated.

The accuracy of the approximate method can be demonstrated by comparison with the rigorous solutions. This can be done by considering a lossless dielectric slab of exponentially varying permittivity. That is

$$\epsilon_r(x) = h \exp.(-x/a),$$

where h is a constant. The rigorous solution of the electric field in the dielectric slab for normal incidence (α = 0) can be expressed in terms of zero order Bessel function.[8]

Table I lists the transmission and reflection coefficients calculated by the method of collocation for $h k_0^2 = (\pi/a)^2$. Only four points are used in this calculation, namely: x = 0, a/3, 2a/3, and a for the even function; x = a/4, a/2, 3a/4, and a for the odd function. These approximate values compared with the exact values show good agreement.

Table I. Comparison of the transmission and reflection coefficients calculated by the method of collocation and the exact values.

	$T \exp(-jk_0 a)$	$R \exp(-2jk_0 a)$
Method of Collocation	$0.9161 \angle 220°5'$	$0.3802 \angle -96°40'$
Exact	$0.9209 \angle 220°23'$	$0.3795 \angle -87°51'$

b. Radiation from a nonuniform dielectric coated sources.

In section 2-a, the method of collocation was applied to the case where the source is at infinity. That this method is applicable to sources near or inside a nonuniform dielectric will be shown as follows.

The geometry under consideration is the same as in section 2-a except that an infinite perfect conducting plane is located at x = 0. Assuming a z-oriented constant phase magnetic line current at x = 0, y = 0, i.e., Eq. (0,y) = Vδ(y), where V, in volts, is a constant, and δ (y) is the Dirac delta function, the problem is then confined to two dimensions. Since the magnetic field has no x- and y-components, the z-component is given by

$$\nabla_\perp^2 H_z - [\ln \epsilon(x)]' \partial_x H_z + k_0^2 \epsilon_r(x) H_z = 0, \qquad (16)$$

where $\nabla_\perp^2$ is the two dimensional Laplacian operator, ∂_x denotes the partial derivative with respect to x. The y-component of electric field is given by

$$E_y = [j\omega\epsilon(x)]^{-1} \partial_x H_z \, . \qquad (17)$$

Introducing the Fourier Transformation

$$H_z(x,y) = \int \bar{H}_z(x,\beta)\exp.(j\beta y)\,d\beta, \tag{18a}$$

$$\bar{H}_z(x,\beta) = \frac{1}{2\pi}\int H_z(x,y)\exp.(-j\beta y)\,dy, \tag{18b}$$

where the integral is from $-\infty$ to ∞, β is the transformation variable. Substituting Eq. (18a) into Eq. (16) yields

$$\partial_x^2 \bar{H}_z(x,\beta) - [\ln \epsilon(x)]'\,\partial_x \bar{H}_z(x,\beta) + [k_0^2\epsilon_r(x) - \beta^2]\bar{H}_z(x,\beta) = 0, \tag{19}$$

From Eqs. (18) and (19), the z-component of magnetic fields in air and inside the dielectric slab may be expressed respectively by

$$H_z^{I}(x,y) = \int C(\beta)\exp.[j(\beta y - \beta_x x)]\,d\beta,$$

$$H_z^{II}(x,y) = \int [A(\beta)U_e(x,\beta) + B(\beta)U_0(x,\beta)]\exp.(j\beta y)\,d\beta,$$

where $\beta_x^2 = k_0^2 - \beta^2$, U_e and U_o are the two linearly independent particular solutions of Eq. (19), A, B, and C, functions of β, are determined by boundary conditions, namely:

$$E_y(0,y) = V\delta(y) = \frac{V}{2\pi}\int \exp.(j\beta y)\,d\beta,$$

$$H_z^{I}(a,y) = H_z^{II}(a,y),$$

$$E_y^{I}(a,y) = E_y^{II}(a,y).$$

Taking the boundary conditions into account, in a straight forward manner, the magnetic field in air is given by

$$H_z^{I}(x,y) = \int F(\beta)\exp.[j(\beta y - \beta_x x)]\,d\beta, \tag{20}$$

where

$$F(\beta) = \frac{j\omega\epsilon(0)}{2\pi} V \frac{\exp(j\beta_x a)}{\partial_x U_e(0,\beta)} \frac{U_e(a,\beta)\,\partial_a U_e(a,\beta) - \partial_a U_o(a,\beta)\,U_e(a,\beta)}{\partial_a U_e(a,\beta) + j\beta_x \epsilon_r(a) U_e(a,\beta)} .$$

For far field considerations, i.e., $k_o\rho \gg 1$, where $x = \rho \cos\theta$, and $y = \rho \sin\theta$, the integral of Eq. (20) can be evaluated approximately by the method of steepest descent.[10] The result is

$$H_z^I(x_o, y_o) \approx (j2\pi k_o\rho_o)^{1/2} F(-k_o \sin\theta_o) \exp(-jk_o\rho_o), \tag{21}$$

where the coordinates without subscript are replaced by two coordinates with subscript o. This is convenient to denote the observation position in the following considerations. Now, the remaining problem is to find two particular solutions U_e and U_o. Substituting $\beta = -k_o \sin\theta_o$ into Eq. (19), one obtains the standard form of Eq. (8) (regarding θ_o as a constant). Using the same procedures outlined in section 2-a Eq. (21) leads to the solution of far field radiation. If a high speed computer is adopted to solve the problem a routine can be set for solutions of Eq. (12). For every specific angle, run the routine once. In this manner, the radiation pattern can be plotted.

Another solvable example of this type is considered next. Consider the same geometry, but let a z-oriented magnetic current sheet on the conducting wall vary periodically in the y-direction. Under these assumptions, the magnetic fields inside and outside the dielectric slab may be written as

$$H_z^{\mathrm{I}, \mathrm{II}} = \sum_{n=-\infty}^{\infty} U_n^{\mathrm{I}, \mathrm{II}}(x) \exp(jn\pi y/L), \tag{22}$$

where 2L is the period of the magnetic current sheet. In air

$$U_n^I(x) = C_n \exp(-jk_n x),$$

where $k_n^2 = k_o^2 - (n\pi/L)^2.$

For the magnetic field inside the dielectric slab, the function U_n^{II} must satisfy the following differential equation:

$$U_n''(x) - [\ln \epsilon(x)]' U_n'(x) + [k_o^2 \epsilon_r(x) - (n\pi/L)^2] U_n(x) = 0. \tag{23}$$

The general solution of Eq. (23) may be represented by

$$U_n(x) = A_n U_{ne}(x) + B_n U_{no}(x).$$

The magnetic sheet current on the conducting wall is characterized by

$$E_y(0, y) = V g(y) = \sum_{n=-\infty}^{\infty} \alpha_n \exp.(jn\pi y/L),$$

where

$$\alpha_n = \frac{V}{2L}\int_{-L}^{L} g(y) \exp.(-jn\pi y/L)\, dy.$$

The y-component of electric fields in air and inside the dielectric slab can be obtained by Eq. (17). Equating the tangential field components at both surfaces of the slab and solving for the expansion coefficients C_n of the magnetic field in air yields

where

$$H_z^r = \sum C_n \exp.(-jk_n x + jn\pi y/L) \tag{24}$$

$$C_n = \frac{j\omega\epsilon(0)}{U_{no}(0)}\alpha_n \frac{U_{no}(a)U'_{ne}(a) - U'_{no}(a)U_{ne}(a)}{U'_{ne}(a) + jk_n\epsilon_r(a)U_{ne}(a)} \exp.(jk_n a).$$

The outside field at any location can be evaluated by Eq. (24) provided that the series of Eq. (24) is uniformly convergent, and the general solution of Eq. (23) is known. The convergence of the series depends on the Fourier representation of the current source. The approximate general solution of Eq. (23) for each n may be obtained by the method of collocation as outlined in section 2-a. Note that for far field considerations, the contribution from the terms with k_n imaginary are negligible, i.e., the terms with $(n\pi/L)^2 > k_0^2$ are omitted from Eq. (24) for the radiation pattern. This method is especially convenient for very low frequencies where $(\pi/L)^2 > k_0^2$ only one term remains in the consideration of the far field radiation.

c. Scattering of a plane wave by cylindrically symmetric nonuniform dielectric cylinder

In previous discussions, the applications of the collocational method are limited to plane geometry. Now, the applications are extended to the cylindrical coordinates and will be formulated as follows:

The geometry considered first is a dielectric cylinder of radius a embedded in free space. The axes of the cylinder is colinear with the z-axis as shown in Figure 2. The relative permittivity of the dielectric is a function of radius only, i.e., $\epsilon_r(\rho)$, for $0 \leq \rho \leq a$, where ρ is the radial coordinate. The two cases of oblique incidence will be considered, namely, the perpendicular polarization and the parallel polarization. The z-component of the incident fields in both cases are respectively given by

$$H_{1z}^{i} = (\cos\alpha/Z_0)\ \exp.(jk_x x + jk_z z), \tag{25a}$$

$$E_{\alpha z}^{i} = (\cos\alpha)\ \exp.(jk_x x + jk_z z), \tag{25b}$$

where the propagation constants k_x and k_z, and the wave impedance Z_o are given as before. The angle α is the angle of incidence. Using the wave transfomation[11], the factor exp. $(jk_x X)$ can be expressed in terms of Bessel functions of the first kind and the cosine function as

$$\exp.(jk_x x) = J_0(\xi) + 2\sum_{n=1}^{\infty} j^n J_n(\xi) \cos\theta, \tag{26}$$

where $x = \rho \cos\theta$, the argument of the Bessel functions is

$$\xi = \rho k_0 \cos\alpha.$$

Since the cylinder is assumed to have infinite length and the permittivity is uniform in the z-direction, the resultant field must be periodic in the z-direction and vary according to the factor exp. $(jk_z z)$. The total fields (incident plus scattering) in air may be written as

$$H_{1z}^{a} = (\cos\alpha/Z_0)\Big\{ J_0(\xi) + a_0 H_0(\xi) + 2\sum_{n=1}^{\infty}\left[j^n J_n(\xi) + a_n H_n(\xi)\right]\cos n\theta\Big\}\exp.(jk_z z), \tag{27}$$

$$E_{\alpha z}^{a} = (\cos\alpha)\Big\{ J_0(\xi) + b_0 H_0(\xi) + 2\sum_{n=1}^{\infty}\left[j^n J_n(\xi) + b_n H_n(\xi)\right]\cos n\theta\Big\}\exp.(jk_z z), \tag{28}$$

where the function $H_n(\xi)$ is the n^{th} order Hankel function of the second kind. For simplicity, the superscript (2) is omitted. The scattering amplitudes a_n and b_n, are determined by boundary conditions. The corresponding fields inside the dielectric cylinder are given by

$$H_{1z}^{d} = \exp.(jk_z z)\sum_{n=0}^{\infty} c_n \bar{\Psi}_n(\rho)\cos n\theta, \tag{29}$$

$$E_{2z}^{d} = \exp.(jk_z z)\sum_{n=0}^{\infty} d_n \bar{\Phi}_n(\rho)\cos n\theta, \tag{30}$$

where the functions $\bar{\Psi}_n$ and $\bar{\Phi}_n$ are the regular particular solutions of the following differential equations respectively

$$\bar{\Psi}_n''(\rho) + [\rho^{-1} - \epsilon_r'(\rho)/W(\rho)]\bar{\Psi}_n'(\rho) + [k_0^2 W(\rho) - n^2\rho^{-2}]\bar{\Psi}_n(\rho) = 0, \tag{31}$$

$$\bar{\Phi}_n''(\rho) + \{\rho^{-1} - [\ln\epsilon(\rho)]' + \epsilon_r'(\rho)/W(\rho)\}\bar{\Phi}_n'(\rho) + [k_0^2 W(\rho) - n^2\rho^{-2}]\bar{\Phi}_n'(\rho) = 0, \tag{32}$$

where $W(\rho) = \epsilon_r(\rho) - \sin^2\alpha$. With knowledge of the z-component of the magnetic field or electric field, the θ-component of the corresponding electric and magnetic fields can be obtained by

$$E_{1\theta} = [j\omega\mu_0/k_0^2 W(\rho)]\,\partial_\rho H_{1z}, \tag{33a}$$

$$H_{2\theta} = [-j\omega\epsilon/k_0^2 W(\rho)]\,\partial_\rho E_{2z} \tag{33b}$$

respectively. By equating the tangential components of the fields at the surface of the cylinder and putting terms of the same angular variation equal to zero, the resultant algebraic equations can be solved for the scattering amplitudes a_n and b_n for each n, namely

$$a_n = j^n \frac{\cos\alpha\,\bar{\Psi}_n'(a)J_n(\xi_0) - k_0 W(a)\,\bar{\Psi}_n(a)\,J_n'(\xi_0)}{k_0 W(a)\,\bar{\Psi}_n(a)H_n'(\xi_0) - \bar{\Psi}_n'(a)H_n(\xi_0)\cos\alpha}, \tag{34a}$$

$$b_n = j^n \frac{\epsilon_r(a)\, \Phi_n'(a) J_n(\xi_c)\cos\alpha - k_o W(a)\, \Phi_n(a) J_n'(\xi_c)}{k_o W(a)\, \Phi_n(a) H_n'(\xi_o) - \epsilon_r(a)\, \Phi_n'(a) H_n(\xi_o)\cos\alpha}, \tag{34b}$$

where $\xi_c = k_o a \cos\alpha$. The solution will be complete if the functions Ψ_n and Φ_n are known. Observe that the differential equations (31) and (32) are of the same form and may be written as

$$V_n''(\rho) + [\rho^{-1} + p(\rho)]\, V_n'(\rho) + [q(\rho) - n^2\rho^{-2}]\, V_n(\rho) = 0, \tag{35}$$

where p (ρ) and q (ρ) are regular functions within the region $0 \leq \rho \leq a$. Analogous to Eq. (10) of section 2-a, the regular particular solution of Eq. (35) may be approximated by

$$V(\rho) = \sum_{m=1}^{M} C_n\, J_n(\alpha_m \zeta), \tag{36}$$

where $\zeta = \rho/ua$ and $J_n(\alpha_m) = 0$, M is an integer. The subscript n of $V_n(\rho)$, C_{nm} and α_{nm} are omitted. The dimensionless parameter u, to be determined by Eq. (35), is a real number which is greater than or equal to unity. Substituting Eq. (36) into (35) yields

$$\sum_{m=0}^{M} \left\{ \frac{\alpha_m}{ua} p(\rho) J_n'(\alpha_m \zeta) + [q(\rho) - \alpha_m^2/(ua)^2] J_n(\alpha_m \zeta) \right\} C_m = 0, \tag{37}$$

Observe that Eq. (37) is similar to Eq. (11). The same procedures outlined in section 2-a may be used to determine the suitable value of u and the expansion coefficients C_m. In other words, if all the sine functions are replaced by J_n, cosine functions by J'_n, $m\pi/v_c$ by α_m/u, and B_m by C_m in Eqs. (11) - (15), then all these equations and the associated discussions are valid for obtaining the regular solution of Eq. (35).

d. Wave propagation in the presence of a cylindrical symmetric nonuniformly coated conducting cylinder.

An important geometry being considered in this section is a conducting cylinder of radius b coated by a cylindrically symmetric nonuniform dielectric medium to radius a as shown in Figure 3. The scattering of a plane wave by this object can be analyzed in the same manner as in section 2-c. Considering again the oblique incidence with perpendicular or parallel polarization, the incident fields in both cases are given by Eqs. (25) of section 2-c. Eqs. (25) through (35) and the associated discussions are valid for the present case except that the two functions Ψ_n and Φ_n are the general solutions of Eqs. (31) and (32) respectively, and the additional boundary conditions.

$$\Psi_n'(b) = 0, \qquad \text{for perpendicular polarization}$$

$$\Phi_n(b) = 0 \qquad \text{for parallel polarization.}$$

are necessary. In this case, since the singular point is not included in the region $b \le \rho \le a$, Eq. (35) can be solved by expressing the solution in terms of trigonometric functions. Substituting $\rho = x + b$ into Eq. (35) yields

$$V_n''(x) + P(x)V_n'(x) + Q_n(x)V_n(x) = 0, \tag{38}$$

where $P(x) = (x+b)^{-1} + p(x+b),$

$Q_n(x) = q(x+b) - n^2/(x+b)^2.$

Eq. (38) is similar to Eq. (8) for the interval $0 \le x \le d$, where d = a - b. Hence, the general solution inside the interval $0 \le x \le d$ can be obtained by using the same procedures as for solving Eq. (8). In other words, if p (x) is replaced by P (x), q (x) by $Q_n(x)$, U (x) by $V_n(x)$ and a by d in Eqs. (8) - (15), then all these equations and the associated statements are valid for obtaining the approximate general solution of Eq. (38). It should be noted that the series of Eqs. (27) and (28) may converge very slowly at high frequencies.

In the presence of a z-directed constant magnetic phase current sheet on the surface of the conducting cylinder, for no z variation, the magnetic field has only a z-component. Inside the dielectric medium, the magnetic field is given by

$$H_z^d = \sum_n \Psi_n(\rho)\exp.(jn\theta),$$

where $\bar{\Psi}_n(\rho)$ is the general solution of Eq. (31) with $\alpha = 0$. The magnetic current sheet may be expressed as

$$E_\theta(b,\theta) = \frac{V}{b} f(\theta) = \sum_n \Upsilon_n \exp(jn\theta)$$

where $$\Upsilon_n = \frac{V}{2\pi b}\int_0^{2\pi} f(\theta)\exp.(-jn\theta)d\theta.$$

The z-component of the magnetic field in air may be written as

$$H_z^a = \sum_n C_n H_n(k_o\rho)\exp.(jn\theta).$$

The θ-component of all electric fields can be obtained by Eq. (33a) with $\alpha = 0$.

The boundary conditions at $\rho = b$ and at $\rho = a$ require

$$C_n = -j\omega\epsilon(b)\Upsilon_n[R'_n(a)S_n(a) - S'_n(a)R_n(a)]/D_n, \quad (40)$$

where $$D_n = H_n(k_o a)[R'_n(a)S'_n(b) - R'_n(b)S'_n(a)] + \epsilon_r(a)k_o H'_n(k_o a)[R'_n(b)S_n(a) - R_n(a)S'_n(b)],$$

$$\bar{\Psi}_n(\rho) = A_n R_n(\rho) + B_n S_n(\rho),$$

A_n and B_n are constants, $R_n(\rho)$ and $S_n(\rho)$ are two linearly independent solutions of Eq. (31) with $\alpha = 0$, inside the closed interval $b \le \rho \le a$. These two functions can be obtained by the method used in solving Eq. (38). The convergence of Eq. (39) depends on the operating frequency and the Fourier representation of the current source.

3. Method of Green's Function

The single spatial variation model is not adequate to represent a re-entering space vehicle in many cases. This can be seen from Figures 8-10 of reference (2). Solutions for wave propagation in a general nonuniform medium (the permittivity and permeability are functions of two or three spatial coordinates) are desirable for this application. It is possible to consider the stimulated polarization of the nonuniformity of the permittivity and the permeability as sources, though these non-linear sources are functions of field strength. Under this consideration, the dyadic Green's function is applicable to formulate the radiation field of the stimulated sources. The integral equation can be solved approximately by iterative method for small variations. The Born approximation is the first order of the iterative approximation.

a. Formulation:

Suppose there exist electric and magnetic sources in a nonuniform medium. The Maxwell equations for time-harmonic varying fields [exp (jωt)] take the following forms:

$$\nabla \times H = j\omega\epsilon E + J,$$
$$\nabla \times E = -j\omega\mu H - J_m,$$
$$\nabla\cdot(\mu H) = \rho_m$$
$$\nabla\cdot(\epsilon E) = \rho_e$$

where the capital letters without subscripts represent vector quantities. E, J, and ρ_e are the electric field strength, current density and charge density; H, i_m and ρ_m are the magnetic field strength, magnetic current density and magnetic charge density respectively. Eliminating E or H from these equations yields

$$\nabla^2 H + k_0^2 H = M, \tag{41}$$

$$\nabla^2 E + k_0^2 E = N, \tag{42}$$

where

$$M = j\omega\epsilon J_m - \nabla\times J - \Delta k^2 H + \nabla(\rho_m/\mu) + \frac{\nabla\epsilon}{\epsilon}\times(J - \nabla\times H) - \nabla\left(\frac{\nabla\mu}{\mu}\cdot H\right), \tag{43}$$

$$N = j\omega\mu J + \nabla\times J_m - \Delta k^2 E + (\rho_e/\mu) - \frac{\nabla\mu}{\mu}\times(J_m + \nabla\times E) - \nabla\left(\frac{\nabla\epsilon}{\epsilon}\cdot E\right). \tag{44}$$

$$\Delta k^2 = k_0^2(f + g + fg),$$

$$\epsilon = \epsilon_0(1+f), \qquad \mu = \mu_0(1+g).$$

Currents and charges of both electric and magnetic types are related by the equations of continuity,

$$\nabla \cdot J + j\omega \rho_e = 0,$$

$$\nabla \cdot J_m + j\omega \rho_m = 0.$$

In most cases, it is difficult to find the solution of Eqs. (41) and (42). Other forms of wave equations are then desirable. Assume that the dynadic Green's function $\bar{G}(R \mid R_o)$ satisfies the inhomogeneous dynadic equation

$$\nabla^2 \bar{G}(R|R_o) + k_o^2 \bar{G}(R|R_o) = -\delta(R-R_o)\bar{I} \tag{45}$$

where $\bar{I}$ is the unity dynadic, i.e., for any vector function F, $\bar{I} \cdot F = F \cdot \bar{I} = F$; and

$$R = \hat{i}x + \hat{j}y + \hat{k}z.$$

Upon multiplying Eq. (45) on the left by H and (41) on the rifht by $\bar{G}$, subtracting the resulting equations and integrating over a volume V enclosed by surface S, one obtains the vector equation

$$\begin{aligned} &\int_V [H \cdot \nabla^2 \bar{G} - (\nabla^2 H) \cdot \bar{G}]\, dV \\ &\quad = -\int_V \delta(R-R_o) H \cdot \bar{I}\, dV - \int_V M \cdot \bar{G}\, dV \end{aligned} \tag{46}$$

Applying the dynadic Green's theorem[11,12] to the left hand side of Eq. (46) and rearranging terms yield

$$\begin{aligned} H(R_o) = -\int_V M \cdot \bar{G}(R|R_o)\, dV - \oint_S n \cdot [&H \times (\nabla \times \bar{G}) + (\nabla \times H) \times G \\ &+ H(\nabla \cdot \bar{G}) - (\nabla \cdot H)\bar{G}]\, dS, \end{aligned} \tag{47}$$

where n is the unit vector outward normal to the surface. The integral equation for the electric strength can be obtained simply by replacing H by E, and M by N in Eq. (47). That is

$$E(R_0) = -\int_V N\cdot \bar{G}(R|R_0)dV - \oint_S n\cdot[E\times(\nabla\times\bar{G}) + (\nabla\times E)\times\bar{G}$$

$$+ E(\nabla\cdot\bar{G}) - (\nabla\cdot E)\bar{G}]dS. \quad (48)$$

The Green's functions of Eqs. (47) and (48) satisfies the same boundary conditions of the magnetic field strength H and the electric field strength E respectively. Hence they may be different. For an infinite domain with specification of outgoing waves at infinity, they are equal and may be written as

$$\bar{G}(R|R_0) = \bar{I}\,\phi(R|R_0) \quad (49)$$

where

$$\phi_2(R|R_0) = \frac{j}{4} H_0^{(2)}(k_0|R-R_0|) \quad \text{for two dimensional problem}$$

$$\phi_3(R|R_0) = e^{jk_0|R-R_0|}/4\pi|R-R_0| \quad \text{for the three dimensional problem,}$$

and the two functions f and g are vanished at infinity. Physically, the problem is considered for the uniform medium with additional sources, $j\omega\mu_0 g H$ and $j\omega\epsilon_0 f E$. Substituting Eq. (49) into Eqs. (47) and (48) and taking into consideration that

$$\nabla\cdot\bar{G} = \nabla\phi,$$
$$A\cdot(\nabla\times\bar{G}) = A\times\nabla\phi,$$
$$n\cdot A\times\bar{G} = (n\times A)\cdot\bar{G},$$

where A is an arbitrary vector, yields

$$
\begin{aligned}
H(R_0) = -\int_V \{ & \phi[j\omega\epsilon J_m - \Delta k^2 H - \tfrac{\nabla\epsilon}{\epsilon} \times (\nabla \times H - J)] \\
& + \nabla\phi \times J - \nabla\phi(\nabla\cdot H) \} dV \\
& - \oint_s [n \times H \times \nabla\phi + (n\cdot H)\nabla\phi + j\omega\epsilon\phi(n\times E)] ds
\end{aligned}
\tag{50}
$$

and

$$
\begin{aligned}
E(R_0) = -\int_V \{ & \phi[j\omega\mu J - \Delta k^2 E - \tfrac{\nabla\mu}{\mu} \times (\nabla \times E + J_m)] \\
& - \nabla\phi \times J_m - \nabla\phi(\nabla\cdot E) \} dV \\
& - \oint_s [n \times E \times \nabla\phi + (n\cdot E)\nabla\phi - j\omega\mu\phi(n\times H)] ds
\end{aligned}
\tag{51}
$$

Rewriting Eqs. (41) and (42) as

$$
\begin{aligned}
\nabla \times \nabla \times H - k_0^2 H = \Delta k^2 H + & \tfrac{\nabla\epsilon}{\epsilon} \times (\nabla \times H - J) \\
& - j\omega\epsilon J_m + \nabla \times J,
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
\nabla \times \nabla \times E - k_0^2 E = \Delta k^2 E + & \tfrac{\nabla\mu}{\mu} \times (\nabla \times E + J_m) \\
& - j\omega\mu J - \nabla \times J_m,
\end{aligned}
\tag{53}
$$

constructing a vector $Q = A\phi$, putting H or E with Q in the Green's second vector identity and following the procedures as outlined by Stratton,[13] gives exactly the same as Eqs. (50) and (51).

b. Scattering of a plane wave by nonuniform media

For an application of the method described previously, the scattering of a plane wave by a nonuniform medium with small variations in ϵ and μ will be considered. The small variation means that f and g, small compared to unity, are continuous functions asymtoptic to zero or equal to zero at infinity. Since the surface S under consideration is receded to infinity, the contributions of the surface integrals vanish. Thus, Eqs. (50) may be written as

$$H(R_0) = H^{in}(R_0) + \int_V \{[\Delta k^2 H + \frac{\nabla\epsilon}{\epsilon} \times \nabla \times H]\phi - (\frac{\nabla\mu}{\mu} \cdot H)\nabla\phi\} dV, \tag{54}$$

where $H = H^s + H^{in}$, and H^{in} and H^s are the H fields of the incident and the scattered wave. By Born-approximation, the scattered-field terms in the integral are neglected. Physically, this means that the incident wave stimulates the radiation from the nonuniform zone without interaction. Under this assumption, Eq. (54) is reduced to be

$$H^s(R_0) = \int_V \{(\Delta k^2 H^{in} + \frac{\nabla\epsilon}{\epsilon} \times \nabla \times H^{in})\phi - \frac{\nabla\mu}{\mu} \cdot H^{in} \nabla\phi\} dV. \tag{55}$$

Confining the nonuniform zone to a finite region, the far field approximation requires that

$$\phi_2 \approx F(R, R_0)(-j/8\pi |R_0|)^{1/2}$$

$$\phi_3 \approx F(R, R_0)/4\pi |R_0|$$

where $$F(R, R_0) = \exp.(-jk_0 |R_0|) \exp.(jk_0 R_0 \cdot R/|R_0|).$$

Eq. (55) can be computed in some simple cases. For example, if ϵ and μ are independent of angular coordinates, the volume integral of Eq. (55) can be evaluated without much difficulty. The electric field strength can be obtained by the same procedures, that is

$$E^s(R_0) = \int_V \{(\Delta k^2 E^{in} + \frac{\nabla\mu}{\mu} \times \nabla \times E^{in})\phi - \frac{\nabla\epsilon}{\epsilon} \cdot E^{in} \nabla\phi\} dV. \tag{56}$$

Usually it is more convenient to utilize the H^s or E^s computed by Eq. (55) or (56) to solve Maxwell's equations for E^s or H^s respectively.

c. Radiation from line source in nonuniform media

The problem is again confined to nonuniform media of finite extent and of simple geometry. The volume and surface integrals of Eqs. (47) and (48) are integrated over this finite region with the dynadic Green's function satisfying

$$\nabla^2 \bar{\bar{G}}(R|R_0) + k_r^2 \bar{\bar{G}}(R|R_0) = -\delta(R-R_0)\bar{\bar{I}}$$

where $k_r^2 = \omega^2 \mu_0 \epsilon_0 \epsilon_r \mu_r$ and ϵ_r and μ_r are the average values of $1+f$ and $1+g$ respectively. Of course, the Green's function satisfies the appropriate boundary conditions. Eqs. (47) and (48) may be solved by approximate methods to evaluate the field intensities at the surface. The space outside the nonuniform zone is free space. The fields strength in this region are solutions with specification of outgoing waves of the uniform wave equation. Their amplitudes can be evaluated by matching the tangential components at the boundary with those calculated by Eqs. (47) and (48) correspondingly. For example, consider a conducting cylinder of radius b and infinite length coated by a dielectric cylindrical ring of radius a. The permittivity is characterized by $\epsilon_0 a^2/r^2$, where r is the radius from the axis. The permeability is constant. The ratio of a to b is equal to 1.2, for this case $\epsilon_r = 1.1934$. Assuming that no current sources exist in the dielectric and free space but an axial magnetic line current $V\delta(\theta)/b$ on the surface of the conducting cylinder, the Green's function is derived by considering an axial magnetic line current $\delta(r-r_0)\,\delta(\theta-\theta_0)/r_0$ in the uniform dielectric ring. That is

$$G(R|R_0) = \frac{\pi}{j}\sum_{n=-\infty}^{\infty}\left\{\begin{bmatrix} H_n'(k_r r_0) J_n(k_r r) \\ J_n(k_r r_0) H_n(k_r r)\end{bmatrix} + A_n J_n(k_r r) + B_n N_n(k_r r)\right\} e^{jn(\theta-\theta_0)} \quad \begin{matrix} r \le r_0 \\ r \ge r_0 \end{matrix} \tag{56}$$

where

$$A_n = D_n^{-1}\{H_n(k_r r_0) J_n'(k_r b)[\sqrt{\epsilon_r}\, H_n'(k_0 a) N_n(k_r a) - H_n(k_0 a) N_n'(k_r a)]$$
$$+ J_n(k_r r) N_n'(k_r b)[H_n(k_0 a) H_n'(k_r a) - \sqrt{\epsilon_r}\, H_n'(k_0 a) H_n(k_r a)]\},$$

$$B_n = D_n^{-1}\{J_n(k_r r_0) J_n'(k_r b)[\sqrt{\epsilon_r}\, H_n'(k_0 a) H_n(k_r a) - H_n(k_0 a) H_n'(k_r a)]$$
$$+ H_n(k_r r_0) J_n'(k_r b)[H_n(k_0 a) J_n'(k_r a) - \sqrt{\epsilon_r}\, J_n(k_r a) H_n'(k_0 a)]\},$$

$$D_n = H_n(k_0 a)[J_n'(k_r b)N_n'(k_r a) - J_n'(k_r a)N_n'(k_r b)]$$
$$+ \sqrt{\epsilon_r}\, H_n'(k_0 a)[J_n(k_r a)N_n'(k_r b)$$

where J_n and N_n are Bessel functions of the first and second kind, H_n is Hankel functionsof the second kind. The z-component of magnetic field strength in the dielectric is given by

$$\begin{aligned} 4\pi H_z^d(R) = & \int_{S_0} G(R|R_0)[k_0^2(a^2/r_0^2 - 1.1934)H_z^d(R_0) - \frac{2}{r_0}\frac{\partial}{\partial r_0} H_z^d(R_0)] dS_0 \\ & + \int_0^{2\pi} [G(R|R_0)\frac{\partial}{\partial r_0} H_z^d(R_0) - H_z^d(R_0)\frac{\partial}{\partial r_0} G(R_0|R)]_{r_0=a} d\theta_0 \\ & + j\omega\epsilon(b) V \sum_{n=-\infty}^{\infty} [a_n J_n(k_r r) + b_n N_n(k_r r)] e^{jn\theta} \end{aligned} \tag{57}$$

where

$$a_n = 2(D_n k_r b)^{-1}[\sqrt{\epsilon_r}\, H_n'(k_0 a)N_n(k_r a) - H_n(k_0 a)N_n'(k_r a)],$$

$$b_n = 2(D_n k_r b)^{-1}[H_n(k_0 a)J_n'(k_r a) - \sqrt{\epsilon_r}\, H_n'(k_0 a)J_n(k_r a)].$$

The magnetic field strength in air may be expressed by

$$H_z^a = \sum_{n=-\infty}^{\infty} C_n H_n(k_0 r) \exp.(jn\theta). \tag{58}$$

Eq. (17) may be solved by iterative method, that is, let

$$H_z^{d(1)} = \frac{j\omega\epsilon_0\epsilon_r}{4\pi} V \sum_{n=-\infty}^{\infty} [a_n J_n(k_r r) + b_n N_n(k_r r)] \exp(jn\theta). \tag{59}$$

Then substituting Eq. (59) into the right hand side of Eq. (57) one obtains $H_z^{d\,(1)}(R)$. Physically, Eq. (59) represents the magnetic field in the dielectric if the permittivity is uniform. $H_z^{d(1)}$ is the first order approximation of the magnetic field or the H field of the Born approximation. Higher order approximations can be obtained accordingly. However, the convergence is not assured for some cases. In Table I, the zero order and the first order approximate values of the expansion coefficient C_o are listed to compare with the exact value at $k_r a = 0.01$.

Table I Approximate and exact values of $C_o / \omega \epsilon_o V$

zero order	first order	exact
0.273	0.236	0.25

Conclusion:

In the foregoing consideration, the method of collocation and the method of least squares were shown applicable to wave propagation through nonuniform regions with variation in only one spatial coordinate. Scattering and radiation in plane and cylindrical cases were formulated by these two methods. Similar analysis will lead to applications of these methods for spherical geometry. The method of collocation has three advantages:

(1) There is no limitation on the variation of permittivity.

(2) Good solutions can be achieved for values of permittivity known only at a few but sufficient points in space.

(3) A closed form approximate expression for fields within the nonuniform region can be obtained.

In cases where the nonuniform region varies in more than one spatial coordinate, Green's function is shown to formulate the integral form of the wave equation. Solution of the integral equation can be obtained by an iterative method for small variations.

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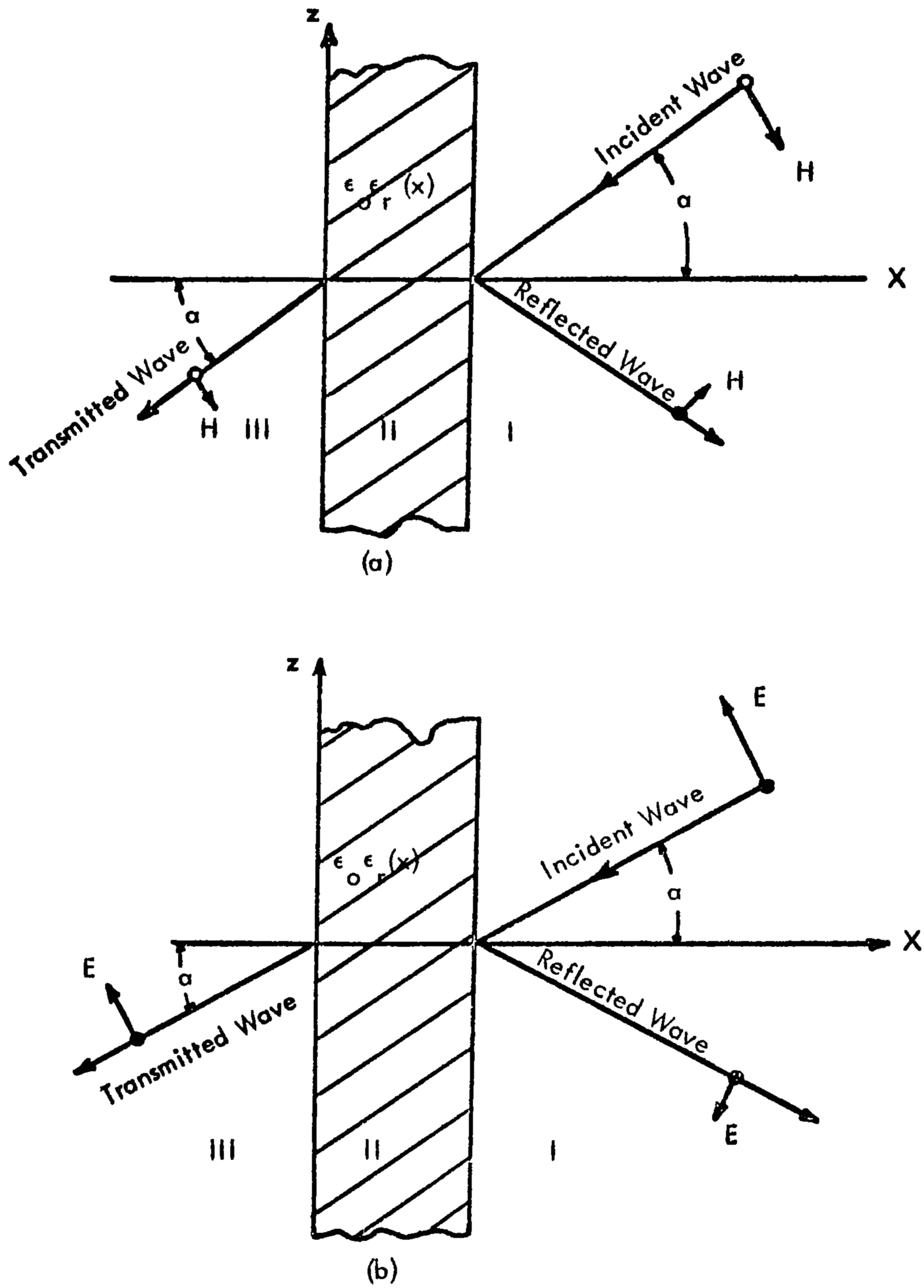

Plane-Sheath Scattering

Figure 1 - (a) Perpendicular polarization.
(b) Parallel polarization.

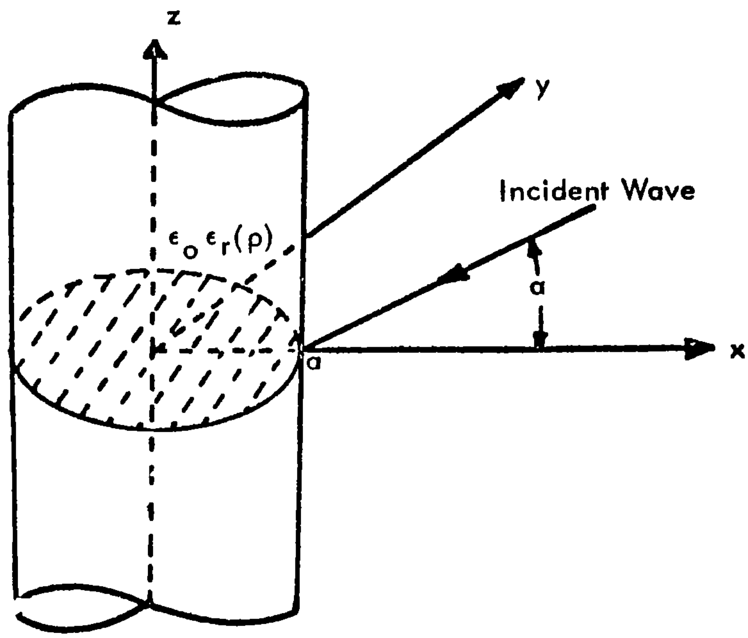

Figure 2 - Scattering of a plane wave by a cylindrical nonuniform dielectric material.

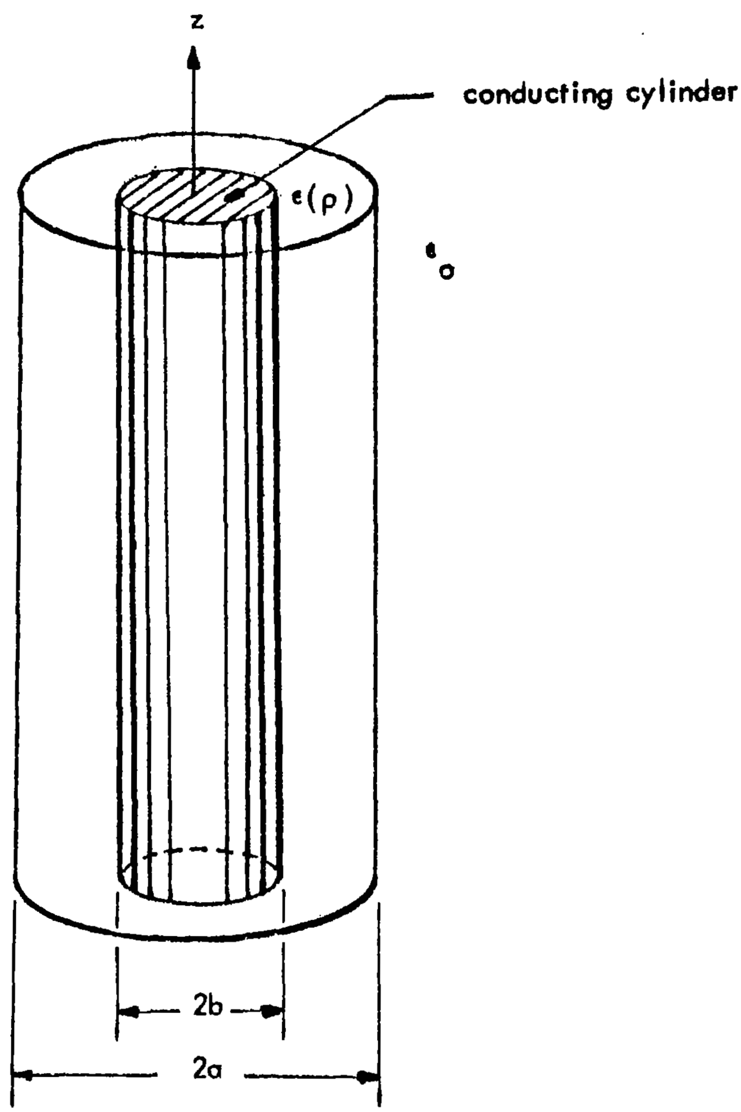

Figure 3 - Conducting cylinder coated by a cylindrically symmetric nonuniform dielectric medium.

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